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Approximate Truncated Balanced Realizations for  
Infinite Dimensional Systems

Tom T. Hartley  
J. Alexis De Abreu - Garcia  
Department of Electrical Engineering  
The University of Akron  
Akron, OH 44325-3904  
216-972-5991  
October 1990

AM 33-09

P-14

### Abstract

This paper presents an approximate method for obtaining truncated balanced realizations of systems represented by non-rational transfer functions, that is infinite dimensional systems. It is based on an approximation to the Hankel operator.

### Acknowledgement

This research was supported by the Advanced Control Technology Branch of NASA Lewis Research Center via NASA Grant NAG 3-904.

### Introduction

The benefits of obtaining accurate low order models are obvious to the control system designer. In particular, if an accurate low order model can be obtained, then a low order controller can be designed which will hopefully maintain the designed closed loop robustness properties when applied to the real system. Much effort has been given to the model reduction problem, particularly in the finite dimensional situation. While input-output methods were prevalent in the 1960's and 1970's [1], the 1980's saw tremendous effort in the state space approach to model reduction [2].

Infinite dimensional systems, on the other hand, have always been of

interest, but have not always been so easy to deal with. Part of this problem comes from the extensive training that the typical control engineer receives in finite dimensional systems via rational polynomial transfer functions. The existence of a non-rational transfer function, other than a time delay operator, is often found to be surprising. Furthermore, the origins of such a transfer function are usually very mysterious. This is probably due to the fact that control engineers typically do not understand spatial boundary value problems. None-the-less, non-rational transfer functions for describing partial differential equations have been in use since Heaviside and the interested reader should consult the wonderful book [3].

Many non-rational transfer functions arise from systems described by wave equations. These will usually have several time-delay operators occupying terms in both the numerator and denominator. Historically, even in the 1950's, these were being approximated by Pade approximations [4]. Here the individual time-delays are replaced by an appropriate order Pade approximation. Then after some tedious algebra, a rational approximation results. The accuracy of the resulting approximation depends significantly on the order of the Pade approximations used with the cost of improved accuracy being increased system order.

Other than some exhausting nonlinear optimization techniques, little significant progress in the reduction of infinite dimensional systems occurred until the late 1980's. This work stemmed from the research in the robust control area and deals with balanced realizations in state space and well as Hankel norms [5]. These methods will be discussed at length in the next section. Several examples will be included to demonstrate the utility of the approximate method developed here.

### Approximate Truncated Balanced Realizations

Truncated balanced realizations for finite dimensional systems are well known and are readily implemented [2]. Many variations are available and the reader is encouraged to consult the large amount of

literature for more information. Recently, Glover et al [5] have been able to develop this technique for infinite dimensional systems. The approach is similar to the finite dimensional situation in that it is based on singular value decomposition. In the finite dimensional case, the balanced realization comes from the leading terms in the singular value decomposition of the necessarily finite rank system Hankel matrix. Similarly, in the infinite dimensional case, the balanced realization is based on an orthogonal expansion (singular value decomposition) of the Hankel operator

$$(\Gamma u)(t) = \int_0^{\infty} h(t+s)u(s)ds$$

where  $h(t)$  is the system impulse response. The difficulty here is that the Hankel operator effectively has infinite rank as nonzero singular values are infinite in number. The real problem in the single-input-single-output case is that this is an integral equation requiring orthogonal expansion of its symmetric kernel,  $h(t+s)$ . This approach is referred to as that of Hilbert-Schmidt. A good discussion of these matters can be found in [6]. This orthogonal expansion for symmetric kernels is based on the eigenvalue-eigenfunction problem

$$u_n(t) = \lambda_n \int_0^{\infty} h(t+s) u_n(s) ds$$

$$v_n(s) = \lambda_n \int_0^{\infty} h(t+s) v_n(t) dt$$

where  $u_n(t)$  and  $v_n(s)$  are the orthogonal eigenfunctions, or Schmidt pairs, corresponding to the eigenvalue  $\lambda_n$ . Once these are found, the Hankel kernel can then be written as

$$h(t+s) = \sum_{n=1}^{\infty} \frac{u_n(t) v_n(s)}{\lambda_n} = \sum_{n=1}^{\infty} \sigma_n u_n(t) v_n(s) .$$

The kernel can then be truncated after the least significant singular value  $\sigma_k$  to yield a rank  $k$  expansion. Once this is done, it is shown

[5] that a state space system of rank  $k$  can be developed which approximates the original system, with error related to  $\sigma_{k+1}$ . There an output normal realization is given and is repeated here;

$$\begin{aligned} B_i &= \sigma_i v_i(0) \\ C_i &= u_i(0) \\ A_{ii} &= -\frac{1}{2} u_i^2(0) \\ A_{ij} &= \frac{B_i^* B_j - \sigma_i^2 C_i^* C_j}{\sigma_i^2 - \sigma_j^2} \end{aligned}$$

Thus, once the singular values and the Schmidt pairs are found, it is then a fairly simple task to obtain a truncated balanced realization. It should also be noted that this reduces to the standard approach for finite dimensional systems. Glover [5] then goes on to give the optimal Hankel norm approximation, but that is not pursued at this time.

The problem with the approach is that it is extremely difficult in most cases to analytically obtain the required orthogonal expansion, that is, the Schmidt pairs. Thus some approximate method for doing this is required. The approach taken by Fredholm in the solution of integral equations [6] is particularly appealing for performing this task. Basically, the Fredholm approach is to divide the integration limits for the Hankel operator into evenly spaced sections in  $s$  and  $t$ , and then to add up the resulting sampled kernels as follows using the rectangular rule for integration;

$$u(s) = \lambda \int_0^\infty h(t+s)u(t)dt$$

then for a symmetric kernel and a timestep  $T$

$$\begin{aligned} u(0) - \lambda T[h(0+0)u(0) + h(0+1)u(1) + \dots + h(0+n)u(n)] &= 0 \\ u(1) - \lambda T[h(1+0)u(0) + h(1+1)u(1) + \dots + h(1+n)u(n)] &= 0 \\ &\vdots \\ u(n) - \lambda T[h(n+0)u(0) + h(n+1)u(1) + \dots + h(n+n)u(n)] &= 0 \end{aligned}$$

Clearly this is still an eigenvalue problem, but rather than finding

Schmidt pairs which are continuous functions of time, the Schmidt pairs will be eigenvectors representing approximate sampled versions of the eigenfunctions. The above equation can be written in matrix form as

$$\begin{bmatrix} 1-\lambda Th_{00} & -\lambda Th_{01} & \cdots & -\lambda Th_{0n} \\ -\lambda Th_{10} & 1-\lambda Th_{11} & \cdots & -\lambda Th_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ -\lambda Th_{n0} & -\lambda Th_{n1} & \cdots & 1-\lambda Th_{nn} \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_n \end{bmatrix} = 0$$

or

$$[I - \lambda TH][u] = 0$$

where H is now the Hankel matrix formed by sampling the system impulse response every T seconds. It should also be remembered that H is symmetric. Thus finding its eigenvalues and eigenvectors is equivalent to a singular value decomposition. Hence, the left and right singular vectors of H will approximate sampled versions of the Schmidt pairs of the orthogonal expansion of the continuous Hankel kernel. Its singular values however are approximations of the singular values obtained from the orthogonal expansion.

Furthermore it should be noted that the singular values of the Hankel matrix must be multiplied by T in order to obtain the above approximation. Since the relative magnitudes should not change, it is then necessary to divide the matrices of right and left singular vectors by SQRT(T). As T gets smaller and smaller, then the approximate version converges to the exact version. The major problem with the approach is in running out of computer memory to form H as T is allowed to get small, however reasonable results can be obtained for fairly large T. Some examples of this are now presented for comparison with the actual solutions.

Example 1: Approximation of the unit triangle impulse response [5,7]. This is included for comparison with the exact solution. Using T=.02 yields a 50x50 Hankel matrix and the following second order system using the output normal realization given above

$$H(s) = \frac{.9778s + 3.4477}{s^2 + 3.6918s + 7.0187}$$

as compared with the actual solution taken from [5]

$$H(s) = \frac{.9561s + 3.6402}{s^2 + 4s + 7.6145}$$

Comparison of the impulse responses is found in Figure 1.

To further demonstrate the process, the solution process for  $T=.2$  is now presented. The sampled Hankel matrix is

$$H = \begin{bmatrix} 1 & .8 & .6 & .4 & .2 \\ .8 & .6 & .4 & .2 & 0 \\ .6 & .4 & .2 & 0 & 0 \\ .4 & .2 & 0 & 0 & 0 \\ .2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The SVD is then performed on this, followed by the T-scaling to give

$$0.2\sigma = [.4036 \ .0647 \ .0247 \ .0145 \ .0109]$$

$$\frac{u}{\sqrt{.2}} = \begin{bmatrix} 1.6321 & 1.1675 & .8121 & .5051 & .2424 \\ 1.2105 & -.2230 & -1.0630 & -1.2828 & -.8422 \\ .8050 & -1.1675 & -.8121 & .7707 & 1.3174 \\ .4435 & -1.3053 & .9079 & .7516 & -1.3078 \\ .1618 & -.7215 & 1.3140 & -1.3930 & .8867 \end{bmatrix}$$

$$\frac{v}{\sqrt{.2}} = [u_1 \ -u_2 \ u_3 \ -u_4 \ u_5].$$

A second order approximation is chosen, then using the output normal realization given above

$$B = \begin{bmatrix} .6587 \\ -.0783 \end{bmatrix}$$

$$C = [1.6321 \ 1.1675]$$

$$A = \begin{bmatrix} -1.3319 & -2.2807 \\ .3752 & -.6815 \end{bmatrix}$$

which is given in input-output form as

$$H(s) = \frac{1.1665s + .8515}{s^2 + 2.0134s + 1.7634}.$$

It can be seen from Figure 2 that the response is not all that bad considering the grossness of the approximation.

Example 2: Approximation of the unit block impulse response [7]. This was done using  $T=.1$ . The resulting system is

$$H(s) = \frac{.8605s + 3.0822}{s^2 + 1.8523s + 3.4681}$$

with corresponding response in Figure 3. Although it doesn't look much like a block, the response is almost identical to that given by [7]. The response from a tenth order approximation is also given and looks pretty good.

Example 3: Approximation of the impulse response  $\sin(t)/t$ . This system is perhaps even more difficult than the others since it is of the infinite impulse response type. Using  $T=.05$  the approximation obtained was

$$H(s) = \frac{1.0169s^2 + 1.0532s + 1.0793}{s^3 + 1.2004s^2 + 1.1813s + .6198}.$$

Although the response is not bad, it did not appear to really be that good, as can be seen from Figure 4. This is probably due to the fact that this system has an infinite impulse response. The reduction of such systems would appear to require a great deal of memory in order to obtain an accurate approximation.

Example 4: Approximation of a simplified supersonic inlet [see 8 for background]. Here the system to be approximated is

$$H(s) = \frac{50 e^{-.015s}}{s + 2 + 50 e^{-.02s}}.$$

The approximate truncated balanced realization is found from an approximate impulse response which is obtained from the following discrete model of this system (it uses z-inverses for the delays and Tustin for the s term),

$$H(z) = \frac{50 (z^6 + z^5)}{2002z^{21} - 1998z^{20} + 50z + 50}, T=0.001.$$

The approximate truncated balanced realization is

$$H(s) = \frac{1000 (-.0025s^2 -.3940s + 523.7)}{s^3 + 105.7s^2 + 8644s + 481446},$$

while a Pade(1,1) approximation for the delays [9] yielded

$$H(s) = \frac{50 (-s^2 + 33.33s + 13333)}{s^3 + 185.3s^2 + 12133s + 693333}.$$

The step responses of each of these systems is plotted in Figure 5 . It can be seen that the approximate truncated balanced realization tracks better initially but that there is a little steady state error. This may be due to either the balanced truncation or the approximation to the Hankel operator. Alternatively, for comparison, the Pade approximation has a much worse transient response, while there is no steady state error. It was also found that higher order truncations of the balanced realization provided more accurate approximations.

## Conclusion

This paper contains an approximate method for obtaining reduced order models of infinite dimensional systems. It is based on an approximation to the Hilbert-Schmidt expansion of the Hankel operator. Much additional work on this approximate approach is necessary. In particular, determining the error bounds associated with this approximation is of primary importance. Furthermore, better approximations to the integral are being considered. Preliminary results indicate that the trapezoidal rule can give a better approximation than the rectangular rule used here. Other methods will also be considered. Another approach is to allow variable values of T for infinite duration impulse responses. Finally, an approach to using frequency domain information or input/output data should be pursued as that is all is available in some cases.



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Figure 1. Truncated balanced approximation for triangular pulse and its approximation

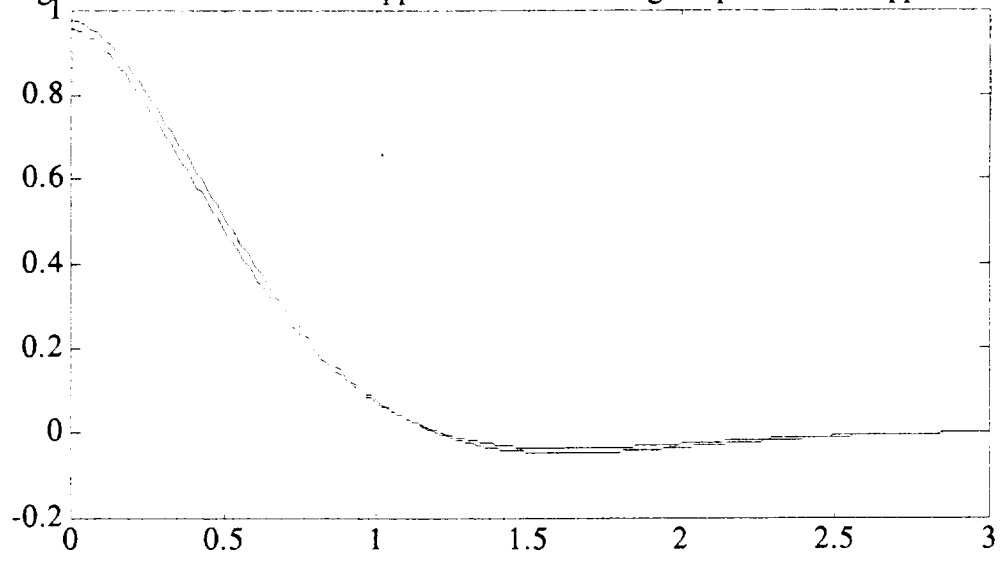


Figure 2. Exact and crude ( $T=0.2$ ) approximation of triangular pulse.

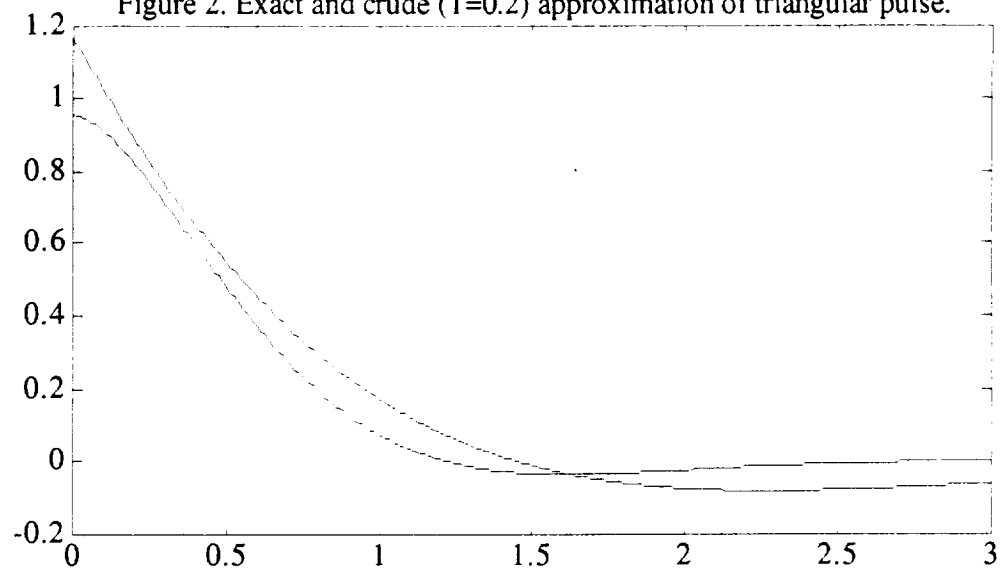


Figure 3. 2nd and 10th order approximations to unit block.

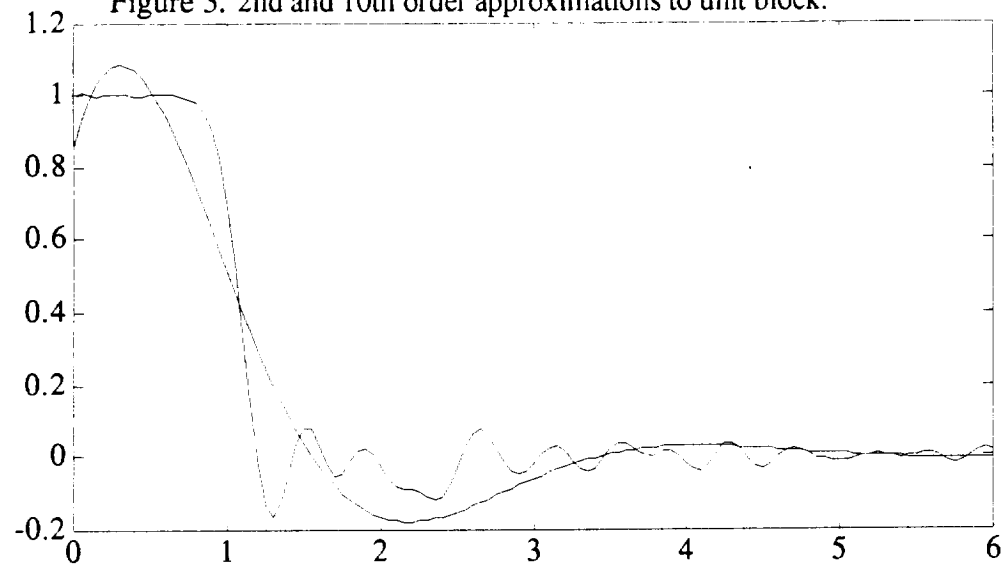


Figure 4. Exact and approximate  $\sin(t)/t$ .

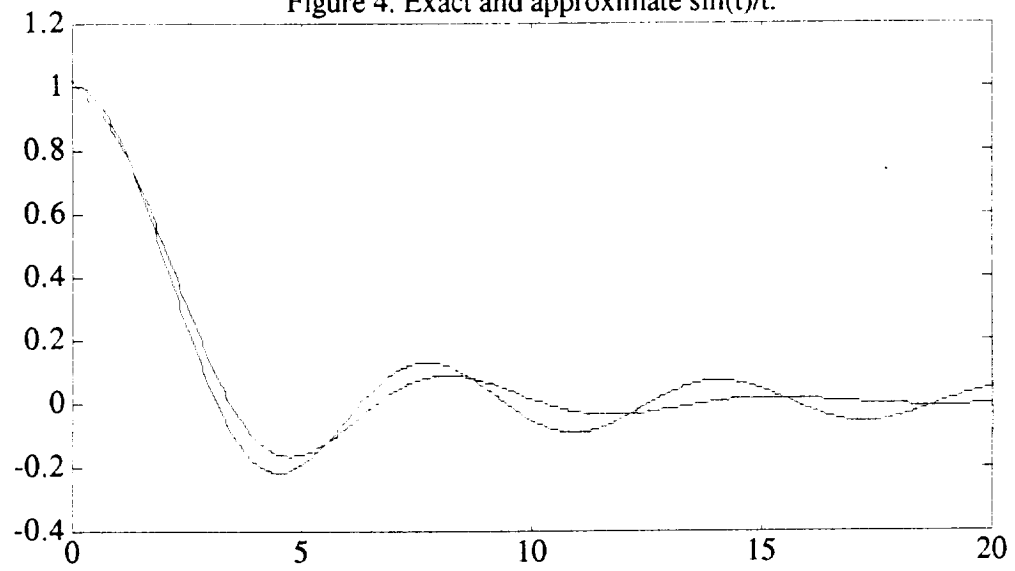


Figure 5. Discrete, Pade, and approximate inlet responses.

